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On cyclic G-designs where G is a cubic tripartite graph

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ABSTRACT

It is known that a ρ-tripartite labeling of a tripartite graph G with n edges can be used to obtain a cyclic G-decomposition of K_{2nt+1} for every positive integer t. We show that if G is an odd prism, an even Möbius ladder or a connected cubic tripartite graph of order at most 10, then G admits a ρ-tripartite labeling. We conjecture that every connected tripartite cubic graph admits a ρ-tripartite labeling.

1. Introduction

If a and b are integers, we denote $\{a, a+1, \dots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_t$ the group of integers modulo t. For a graph G, let rG denote the graph comprised of r vertex-disjoint copies of G. We shall call G *tripartite* if the chromatic number of G is at most 3. Thus, bipartite graphs can be considered tripartite.

Let $V(K_t) = \{0, 1, \dots, t-1\}$. The *length* of an edge $\{i, j\}$ in K_t is $\min\{|i-j|, t-|i-j|\}$. Note that if t is odd, then K_t consists of t edges of length i for $i = 1, 2, \dots, \frac{t-1}{2}$.

Let $V(K_t) = \mathbb{Z}_t$ and let G be a subgraph of K_t. By *clicking* G, we mean applying the permutation $i \to i+1$ to $V(G)$. Note that clicking an edge does not change its length. Let H and G be graphs such that G is a subgraph of H. A G-*decomposition* of H is a set $\Delta = \{G_1, G_2, \dots, G_r\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{r} E(G_i)$. A G-decomposition of K_t is also known as a (K_t, G)-*design*. A (K_t, G)-design Δ is *cyclic* if clicking is an automorphism of Δ. The study of graph decompositions is generally known as the study of graph designs, or G-designs. For recent surveys on G-designs, see [1,2].

Let G be a graph of size n. A primary question in the study of graph designs is: *for what values of k does there exist a (K_k, G)-design*? For most studied graphs G, it is often the case that if $k \equiv 1 \pmod{2n}$, then there exists a (K_k, G)-design. A common approach to finding these designs is through the use of graph labelings.

1.1. Graph labelings

For graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [11], Rosa introduced a hierarchy of labelings. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) =$

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$\{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{f}(E(G)) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

(ℓ1) $f(V(G)) \subseteq [0, 2n]$,
(ℓ2) $f(V(G)) \subseteq [0, n]$,
(ℓ3) $\bar{f}(E(G)) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,
(ℓ4) $\bar{f}(E(G)) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ consider also

(ℓ5) for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,
(ℓ6) there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

(ℓ1), (ℓ3) is called a ρ-*labeling*;
(ℓ1), (ℓ4) is called a σ-*labeling*;
(ℓ2), (ℓ4) is called a β-*labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. Suppose G is bipartite. If a ρ, σ, or β-labeling of G satisfies condition (ℓ5), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition (ℓ6) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an α-*labeling* as introduced in [11]. Labelings of the types above are called *Rosa-type labelings* because of Rosa's original article [11] on the topic. (See [4] for a recent comprehensive survey of Rosa-type labelings.) A dynamic survey on general graph labelings is maintained by Gallian [8].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from [11].

Theorem 1. *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G admits a ρ-labeling.*

Theorem 2. *Let G be a bipartite graph with n edges that admits an α-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

From a graph decompositions perspective, Theorem 2 offers a great advantage over Theorem 1. However, there are many classes of bipartite graphs (see [4]) that do not admit α-labelings. Theorem 2 was extended to cover graphs that admit ρ^+-labelings in [5].

Theorem 3. *Let G be a bipartite graph with n edges that admits a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

Labelings that lead to results similar to those of Theorem 3 have now been introduced for tripartite graphs [3].

Let G be a tripartite graph with n edges having the vertex tripartition $\{A, B, C\}$. A ρ-*tripartite* labeling of G is a one-to-one function $h : V(G) \to [0, 2n]$ that satisfies

(r1) h is a ρ-labeling of G;
(r2) if $\{a, v\} \in E(G)$ with $a \in A$, then $h(a) < h(v)$;
(r3) if $e = \{b, c\} \in E(G)$ with $b \in B$ and $c \in C$, then there exists an edge $e' = \{b', c'\} \in E(G)$ with $b' \in B$ and $c' \in C$ such that $|h(c') - h(b')| + |h(c) - h(b)| = 2n$;
(r4) if $b \in B$ and $c \in C$, then $|h(b) - h(c)| \neq 2n$.

Note that if G is bipartite, then we can take the set C in the tripartition to be empty. In this case, a ρ^+-labeling (and hence an α-labeling) of G is a ρ-tripartite labeling.

The following theorem from [3] shows that a ρ-tripartite labeling yields results similar to α- and ρ^+-labelings.

Theorem 4. *If a tripartite graph G with n edges has a ρ-tripartite labeling, then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

We illustrate how Theorem 4 works. Let G have n edges and let h be a ρ-tripartite labeling for G, with A, B, and C as in the above definition. Let $B_1, B_2, \ldots, B_t$ be t vertex-disjoint copies of B, and let $C_1, C_2, \ldots, C_t$ be t vertex-disjoint copies of C. The vertex in B_i corresponding to $b \in B$ will be called b_i. Similarly, the vertex in C_i corresponding to $c \in C$ will be called c_i. Let $B^* = \bigcup_{i=1}^{t} B_i$ and $C^* = \bigcup_{i=1}^{t} C_i$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \leq i \leq t$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{b_i, c_i\}$, $1 \leq i \leq t$, whenever $\{b, c\}$ is an edge of G with $b \in B$ and $c \in C$. Clearly G^* has nt edges and G decomposes to G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (t-i)2n & v = c_i \in C_i. \end{cases}$$

The labeling h^* is a ρ-labeling of G^* and the result follows by Theorem 1.

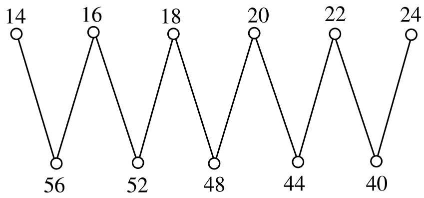


Fig. 1. The path $\hat{P}(10, 2, 4, 14, 40)$.

Some Rosa-type labelings of various cubic graphs have been investigated. It is known that all bipartite prisms [7,6] and bipartite Möbius ladders [9] admit α-labelings. In [14], it is shown that if G is cubic and bipartite and if every component of G is either a prism, a Möbius ladder or has order at most 14, then G admits an α-labeling. In [15], it is shown that every cubic graph of order at most 12, other than $2K_4$ and $3K_4$, admits a β-labeling. Vietri [12,13] has shown that certain classes of generalized Petersen graphs are graceful. It is also known that $2K_4$ does not admit a ρ-labeling, but $3K_4$ does.

In this article, we show that if G is an odd prism, an even Möbius ladder or a connected cubic tripartite graph of order at most 10, then G admits a ρ-tripartite labeling. We conjecture that every connected tripartite cubic graph admits a ρ-tripartite labeling.

2. Additional definitions and notation

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \le i \le k-1$, by $(x_0, x_1, \ldots, x_k)$. In using this notation, we are thinking of traversing the path from x_0 to x_k so that x_0 is the first vertex, x_1 is the second vertex, and so on. Let $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$. If G_1 and G_2 are vertex-disjoint except for $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$. If the only vertices they have in common are $x_0 = y_k$ and $x_j = y_0$, then by $G_1 + G_2$ we mean the cycle $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_{k-1}, x_0)$.

Let $P(2k)$ be the path with $2k$ edges and $2k+1$ vertices $0, 1, \ldots, 2k$ given by $(0, 2k, 1, 2k-1, 2, 2k-2, \ldots, k-1, k+1, k)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, k]$, $B = [k+1, 2k]$, and every edge joins a vertex from A to one from B. Furthermore, the set of labels of the edges of $P(2k)$ is $[1, 2k]$.

Let a and b be nonnegative integers and k, d_1, and d_2 be positive integers such that $a + kd_1 < b$. Let $\hat{P}(2k, d_1, d_2, a, b)$ be the path with $2k$ edges and $2k+1$ vertices given by $(a, b+(k-1)d_2, a+d_1, b+(k-2)d_2, a+2d_1, \ldots, a+(k-1)d_1, b, a+kd_1)$. Note that $\hat{P}(2k, 1, 1, 0, k+1)$ is the graph $P(2k)$. Note that this graph $\hat{P}(2k, d_1, d_2, a, b)$ has the following properties.

P1: $\hat{P}(2k, d_1, d_2, a, b)$ is a path with first vertex a, second vertex $b + (k-1)d_2$, and last vertex $a + kd_1$.
P2: Each edge of $\hat{P}(2k, d_1, d_2, a, b)$ joins a vertex from $A = \{a + id_1 : 0 \le i \le k\}$ to a vertex with a larger label from $B = \{b + id_2 : 0 \le i \le k-1\}$.
P3: The set of edge labels of $\hat{P}(2k, d_1, d_2, a, b)$ is $\{b - a - kd_1 + i(d_1 + d_2) : 0 \le i \le k-1\} \cup \{b - a - (k-1)d_1 + i(d_1 + d_2) : 0 \le i \le k-1\}$.

The path $\hat{P}(10, 2, 4, 14, 40)$ is shown in Fig. 1.

3. ρ-tripartite labelings of some cubic graphs

We will show that odd prisms, even Möbius ladders, and tripartite cubic graphs of order at most 10 admit ρ-tripartite labelings.

3.1. ρ-tripartite labelings of odd prisms

By the *prism* D_n ($n \ge 3$) we mean the Cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices. For convenience, we let $D_n = C_n \cup C'_n \cup F$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$, $C'_n = (v'_1, v'_2, \ldots, v'_n, v'_1)$, and $F = \{\{v_i, v'_i\} : 1 \le i \le n\}$. We shall refer to C_n as the *outer* cycle, to C'_n as the *inner* cycle, and to F as the *spokes*. We note that D_{2n+1}, $n \ge 1$, is necessarily tripartite with tripartition $\{A, B, C\}$ where $A = \{v_1\} \cup \{v_{2i} : 2 \le i \le n\} \cup \{v'_{2i+1} : 1 \le i \le n\}$, $B = \{v_2\} \cup \{v'_1\} \cup \{v_{2i+1} : 2 \le i \le n\} \cup \{v'_{2i} : 2 \le i \le n\}$, and $C = \{v_3, v'_2\}$. Fig. 2 shows the prism D_7. In this figure, the vertices in A are shown with open circles while the vertices in B are shown with filled circles and the vertices of C are shown with open squares. The edges between sets B and C are shown in thick lines. We will adopt this convention in all our figures. We will show that D_n admits a ρ-tripartite labeling for all odd integers $n \ge 3$.

Theorem 5. *The prism D_n admits a ρ-tripartite labeling for all odd $n \ge 3$.*

Proof. We separate the proof into 3 cases.

Case 1 $n \equiv 1 \pmod 6$.

Let $n = 6t + 1$. Thus, $|V(D_n)| = 12t + 2$ and $|E(D_n)| = 18t + 3$. A ρ-tripartite labeling of D_7 is given in Fig. 2.

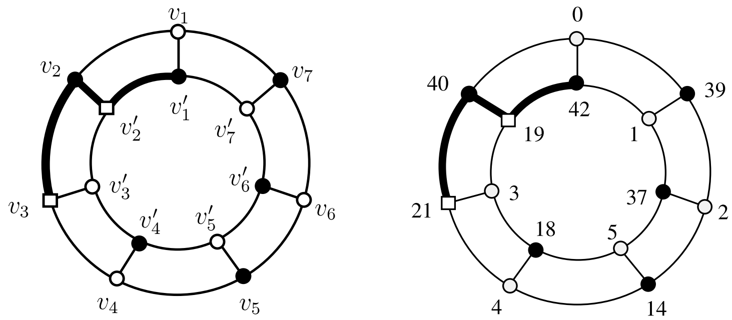


Fig. 2. The prism D_7 and a ρ-tripartite labeling of D_7.

For $t \geq 2$, define a one-to-one function $f : V(D_{6t+1}) \to [0, 36t + 6]$ as follows:

$f(v_1) = 0$,
$f(v_2) = 36t + 4$,
$f(v_3) = 18t + 3$,
$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ even},\ 4 \leq i \leq 6t - 2\}$,
$f(v_i) = 18t - 2i + 8, \quad v_i \in B_1 = \{v_i : i \text{ odd},\ 5 \leq i \leq 4t - 1\}$,
$f(v_i) = 18t - 2i + 6, \quad v_i \in B_2 = \{v_i : i \text{ odd},\ 4t - 1 < i \leq 6t - 1\}$,
$f(v_{6t}) = 2$,
$f(v_{6t+1}) = 36t + 3$,
$f(v_1') = 36t + 6$,
$f(v_2') = 18t + 1$,
$f(v_i') = i, \quad v_i' \in A_1' = \{v_i' : i \text{ odd},\ 3 \leq i \leq 6t - 1\}$,
$f(v_i') = 18t - 2i + 8, \quad v_i' \in B_1' = \{v_i' : i \text{ even},\ 4 \leq i \leq 4t\}$,
$f(v_i') = 18t - 2i + 6, \quad v_i' \in B_2' = \{v_i' : i \text{ even},\ 4t < i \leq 6t - 2\}$,
$f(v_{6t}') = 36t + 1$,
$f(v_{6t+1}') = 1$.

Note that $A = \{v_1, v_{6t}, v_{6t+1}'\} \cup A_1 \cup A_1'$, $B = \{v_2, v_{6t+1}, v_1', v_{6t}'\} \cup B_1 \cup B_2 \cup B_1' \cup B_2'$ and $C = \{v_3, v_2'\}$. Thus the domain of f is indeed $V(D_{6t+1})$. Next, we confirm that f is one-to-one. We compute

$f(A_1) = \{4, 6, \ldots, 6t - 2\}$,
$f(A_1') = \{3, 5, \ldots, 6t - 1\}$,
$f(B_1) = \{18t - 2, 18t - 6, \ldots, 10t + 10\}$,
$f(B_2) = \{10t + 4, 10t, \ldots, 6t + 8\}$,
$f(B_1') = \{18t, 18t - 4, \ldots, 10t + 8\}$,
$f(B_2') = \{10t + 2, 10t - 2, \ldots, 6t + 10\}$.

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) = [0, 6t - 1]$ and $f(B \cup C) \subseteq [6t, 36t + 6]$.

To help compute the edge labels, we will describe $f(V(D_{6t+1}))$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t+1} and C_{6t+1}' with their labels. We have $f(C_{6t+1}) = G_1 + G_2 + (6t - 2, 6t + 8, 2, 36t + 3, 0, 36t + 4, 18t + 3, 4)$, where

$G_1 = \hat{P}(2(2t - 2), 2, 4, 4, 10t + 10)$,
$G_2 = \hat{P}(2(t - 1), 2, 4, 4t, 6t + 12)$.

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{6t + 10 + 6i : 0 \leq i \leq 2t - 3\} \cup \{6t + 12 + 6i : 0 \leq i \leq 2t - 3\} \\ &= \{\ell \equiv 4 \pmod 6 : 6t + 10 \leq \ell \leq 18t - 8\} \cup \{\ell \equiv 0 \pmod 6 : 6t + 12 \leq \ell \leq 18t - 6\}, \\ \bar{f}(E(G_2)) &= \{14 + 6i : 0 \leq i \leq t - 2\} \cup \{16 + 6i : 0 \leq i \leq t - 2\} \\ &= \{\ell \equiv 2 \pmod 6 : 14 \leq \ell \leq 6t + 2\} \cup \{\ell \equiv 4 \pmod 6 : 16 \leq \ell \leq 6t + 4\}. \end{aligned}$$

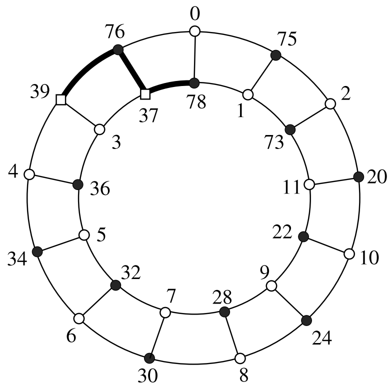


Fig. 3. A ρ-tripartite labeling of D_{13}.

Moreover, edge labels 10, $6t+6$, $36t+1$, $36t+3$, $36t+4$, $18t+1$ and $18t-1$ occur on the path $(6t-2, 6t+8, 2, 36t+3, 0, 36t+4, 18t+3, 4)$.

Similarly, we have $f(C'_{6t+1}) = G'_1 + G'_2 + (6t-1, 36t+1, 1, 36t+6, 18t+1, 3)$, where

$$G'_1 = \hat{P}(2(2t-1), 2, 4, 3, 10t+8),$$
$$G'_2 = \hat{P}(2(t-1), 2, 4, 4t+1, 6t+10).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{6t+7+6i : 0 \le i \le 2t-2\} \cup \{6t+9+6i : 0 \le i \le 2t-2\}\\ &= \{\ell \equiv 1 \pmod 6 : 6t+7 \le \ell \le 18t-5\} \cup \{\ell \equiv 3 \pmod 6 : 6t+9 \le \ell \le 18t-3\},\\ \bar{f}(E(G'_2)) &= \{11+6i : 0 \le i \le t-2\} \cup \{13+6i : 0 \le i \le t-2\}\\ &= \{\ell \equiv 5 \pmod 6 : 11 \le \ell \le 6t-1\} \cup \{\ell \equiv 1 \pmod 6 : 13 \le \ell \le 6t+1\}.\end{aligned}$$

Moreover, edge labels $30t+2$, $36t$, $36t+5$, $18t+5$ and $18t-2$ occur on the path $(6t-1, 36t+1, 1, 36t+6, 18t+1, 3)$.

For each spoke $\{v_i, v'_i\}$, the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t+6 & \text{for } i = 1,\\ 18t+3 & \text{for } i = 2,\\ 18t & \text{for } i = 3,\\ 18t-3i+8 & \text{for } 4 \le i \le 4t,\\ 18t-3i+6 & \text{for } 4t+1 \le i \le 6t-1,\\ 36t-1 & \text{for } i = 6t,\\ 36t+2 & \text{for } i = 6t+1.\end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = &\{\ell \equiv 2 \pmod 3 : 6t+8 \le \ell \le 18t-4\} \cup \{\ell \equiv 0 \pmod 3 : 9 \le \ell \le 6t+3\}\\ &\cup \{36t+6, 18+3, 18t, 36t-1, 36t+2\}.\end{aligned}$$

It is easy to verify now that for each $\ell \in [1, 18t+3]$ either ℓ or $36t+7-\ell$ occurs on exactly one edge in D_{6t+1}. Hence the defined labeling is a ρ-labeling, and condition (r1) for a ρ-tripartite labeling is satisfied. Condition (r2) also holds since $f(A) = [0, 6t-1]$ and $f(B \cup C) \subseteq [6t, 36t+6]$. Condition (r3) holds since $|f(v'_1) - f(v'_2)| + |f(v_2) - f(v_3)| = 36t+6$ and $|f(v_2) - f(v'_2)| + |f(v_2) - f(v'_2)| = 36t+6$, twice the number of edges of D_{6t+1}. Also $|f(b) - f(c)| = 36t+6$, where $b \in B$ and $c \in C$, is impossible since all vertex labels are in $[0, 36t+6]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of D_{6t+1}. Fig. 3 shows a ρ-tripartite labeling of D_{13}.

Case 2 $n \equiv 3 \pmod 6$.

Let $n = 6t-3$. Thus, $|V(D_n)| = 12t-6$ and $|E(D_n)| = 18t-9$. A ρ-tripartite labeling of D_3 is given in Table 1 (graph C2). For $t \ge 2$, Define a one-to-one function $f : V(D_{6t-3}) \to [0, 36t-18]$ as follows:

$$f(v_1) = 0,$$
$$f(v_2) = 36t-20,$$
$$f(v_3) = 18t-9,$$
$$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ even},\ 4 \le i \le 6t-4\},$$
$$f(v_i) = 18t-2i-4, \quad v_i \in B_1 = \{v_i : i \text{ odd},\ 5 \le i \le 4t-3\},$$

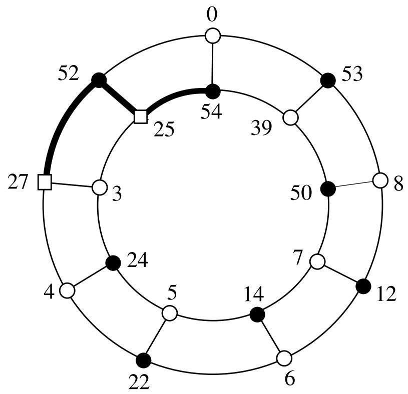


Fig. 4. A ρ-tripartite labeling of D_9.

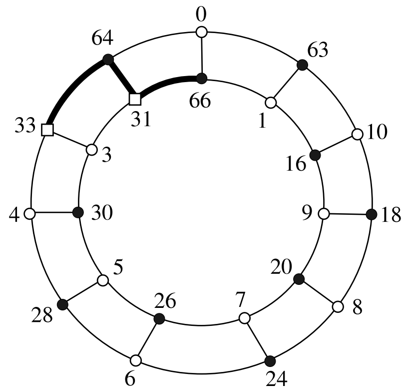


Fig. 5. A ρ-tripartite labeling of D_{11}.

$$
\begin{aligned}
&f(v_i) = 18t - 2i - 10, \quad v_i \in B_2 = \{v_i : i \text{ odd},\ 4t - 3 < i \le 6t - 5\},\\
&f(v_{6t-3}) = 36t - 19,\\
&f(v_1') = 36t - 18,\\
&f(v_2') = 18t - 11,\\
&f(v_i') = i, \quad v_i' \in A_1' = \{v_i' : i \text{ odd},\ 3 \le i \le 6t - 5\},\\
&f(v_i') = 18t - 2i - 4, \quad v_i' \in B_1' = \{v_i' : i \text{ even},\ 4 \le i \le 4t - 4\},\\
&f(v_i') = 18t - 2i - 10, \quad v_i' \in B_2' = \{v_i' : i \text{ even},\ 4t - 4 < i \le 6t - 6\},\\
&f(v_{6t-4}') = 36t - 22,\\
&f(v_{6t-3}') = 30t - 21.
\end{aligned}
$$

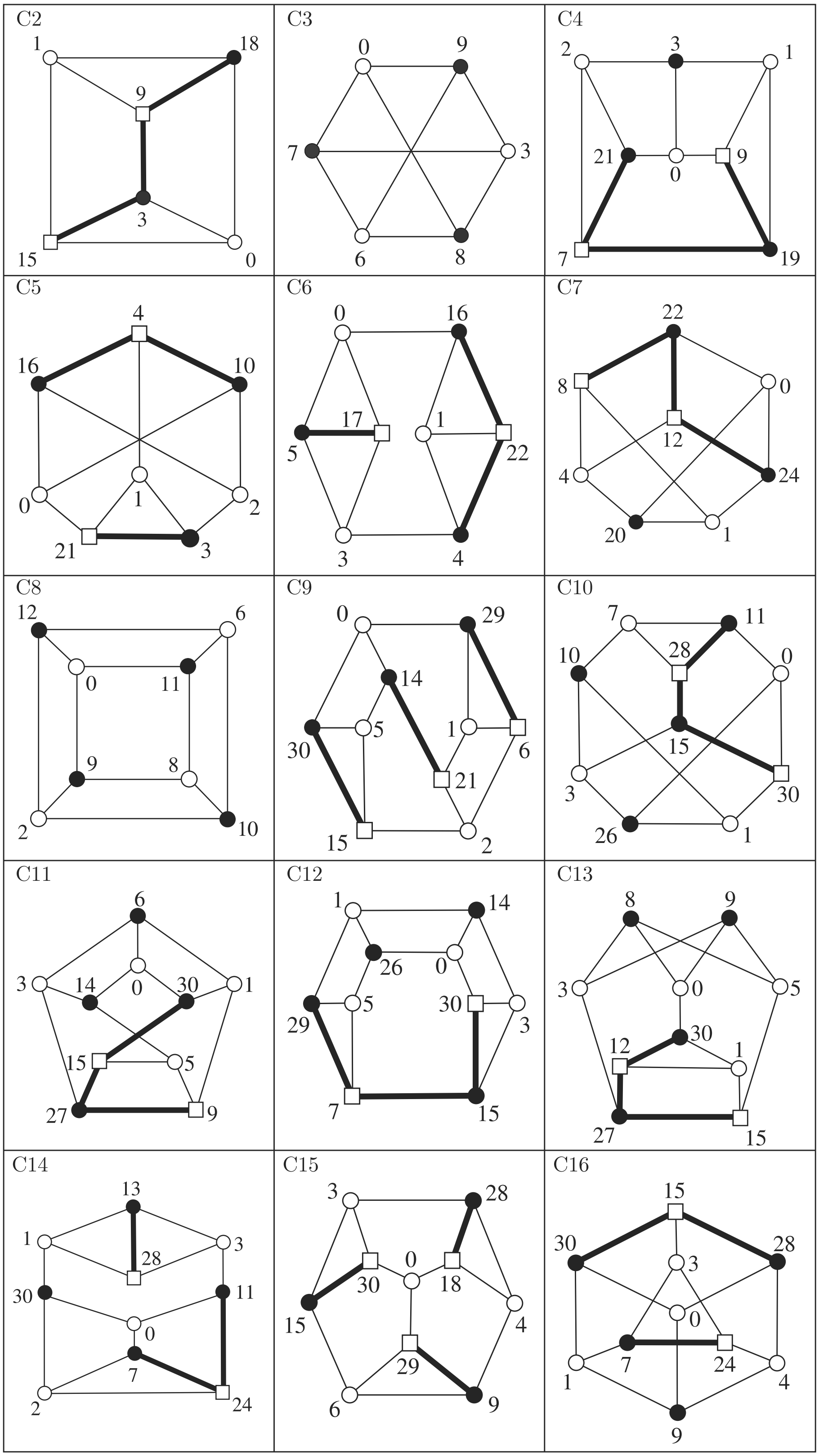
Note that $A = \{v_1, v_{6t-3}'\} \cup A_1 \cup A_1'$, $B = \{v_2, v_1', v_{6t-3}, v_{6t-4}'\} \cup B_1 \cup B_2 \cup B_1' \cup B_2'$ and $C = \{v_3, v_2'\}$. If we proceed as in case 1, it is easy to verify that we have a ρ-tripartite labeling of D_{6t-3}. Fig. 4 shows a ρ-tripartite labeling of D_9.

Case 3 $n \equiv 5 (\text{mod } 6)$.

Let $n = 6t - 1$. Thus, $|V(D_n)| = 12t - 2$ and $|E(D_n)| = 18t - 3$. A ρ-tripartite labeling of D_5 is given in Table 1 (graph C23). For $t \ge 2$, Define a one-to-one function $f : V(D_{6t-1}) \to [0, 36t - 6]$ as follows:

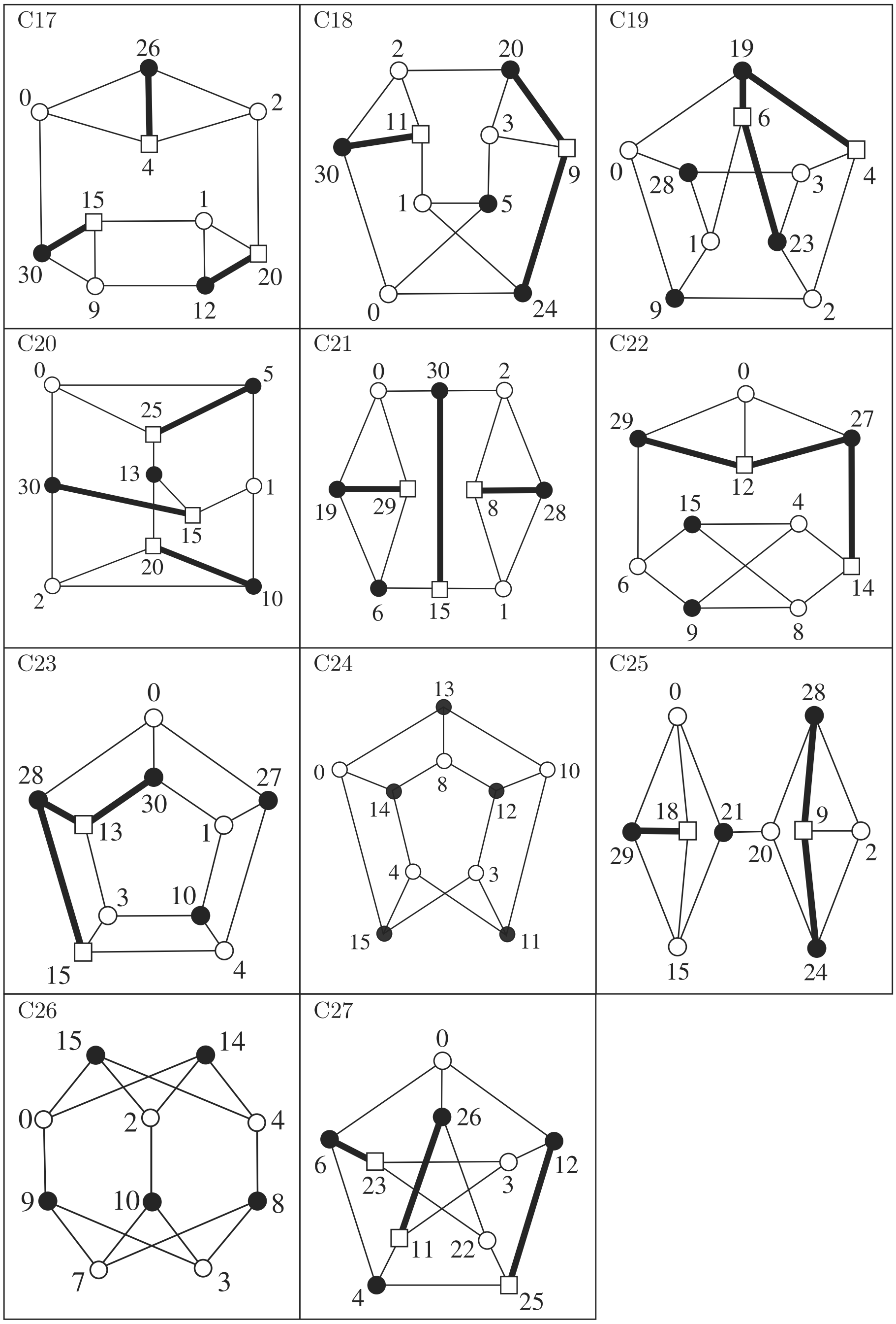
$$
\begin{aligned}
&f(v_1) = 0,\\
&f(v_2) = 36t - 8,\\
&f(v_3) = 18t - 3,\\
&f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ even},\ 4 \le i \le 6t - 2\},\\
&f(v_i) = 18t - 2i + 2, \quad v_i \in B_1 = \{v_i : i \text{ odd},\ 5 \le i \le 4t - 1\},\\
&f(v_i) = 18t - 2i, \quad v_i \in B_2 = \{v_i : i \text{ odd},\ 4t - 1 < i \le 6t - 3\},\\
&f(v_{6t-1}) = 36t - 9,\\
&f(v_1') = 36t - 6,
\end{aligned}
$$

Table 1
ρ-tripartite labelings of tripartite cubic graphs of order ≤ 10.



(*continued on next page*)

Table 1 (*continued*)



$$
\begin{aligned}
&f(v_2') = 18t - 5,\\
&f(v_i') = i, \quad v_i' \in A_1' = \{v_i' : i \text{ odd},\ 3 \le i \le 6t - 3\},\\
&f(v_i') = 18t - 2i + 2, \quad v_i' \in B_1' = \{v_i' : i \text{ even},\ 4 \le i \le 4t - 2\},\\
&f(v_i') = 18t - 2i, \quad v_i' \in B_2' = \{v_i' : i \text{ even},\ 4t - 2 < i \le 6t - 2\},\\
&f(v_{6t-1}') = 1.
\end{aligned}
$$

We have that $A = \{v_1, v_{6t-1}'\} \cup A_1 \cup A_1'$, $B = \{v_2, v_{6t-1}, v_1'\} \cup B_1 \cup B_2 \cup B_1' \cup B_2'$ and $C = \{v_3, v_2'\}$. If we proceed as in case 1, it is easy to verify that we have a ρ-tripartite labeling of D_{6t-1}. Fig. 5 shows a ρ-tripartite labeling of D_{11}. $\square$

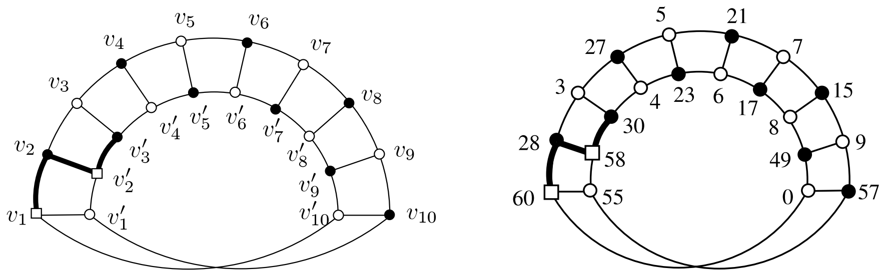


Fig. 6. The Möbius ladder M_{10} and a ρ-tripartite labeling of M_{10}.

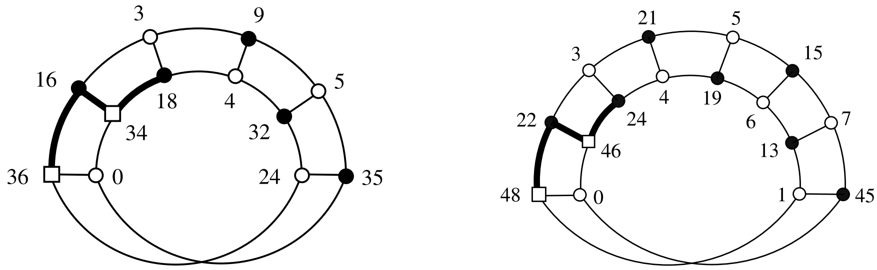


Fig. 7. ρ-tripartite labelings of M_6 and M_8.

3.2. ρ-tripartite labelings of even Möbius ladders

For $n \geq 3$, let $v_1, v_2, \ldots, v_n$ and $v_1', v_2', \ldots, v_n'$ denote the consecutive vertices of two disjoint paths with n vertices. The *Möbius ladder* is the graph M_n obtained by joining v_i to v_i' for $i = 1, 2, \ldots, n$ and by joining v_1 to v_n' and v_n to v_1'. For convenience, we let $M_n = P_n \cup P_n' \cup F \cup H$, where $P_n = (v_1, v_2, \ldots, v_n)$, $P_n' = (v_1', v_2', \ldots, v_n')$, $F = \{\{v_i, v_i'\} : 1 \leq i \leq n\}$ and $H = \{\{v_1, v_n'\}, \{v_n, v_1'\}\}$. We shall refer to P_n as the *outer* path, to P_n' as the *inner* path, and to F as the *spokes*. Fig. 6 shows the Möbius ladder M_{10}. We note that M_{2n}, $n \geq 2$, is necessarily tripartite with tripartition $\{A, B, C\}$, where $A = \{v_1', v_{2i-1} : 2 \leq i \leq n, v_{2i}' : 2 \leq i \leq n\}$, $B = \{v_{2i} : 1 \leq i \leq n, v_{2i-1}' : 2 \leq i \leq n\}$ and $C = \{v_1, v_2'\}$. We will show that M_n admits a ρ-tripartite labeling for all even integers $n \geq 4$.

Lemma 6. *The Möbius ladder M_n admits a ρ-tripartite labeling for all $n \in \{4, 6, 8, 10\}$.*

Proof. A ρ-tripartite labeling of M_4 is given in Table 1 (graph C7). We give ρ-tripartite labelings of M_6 and M_8 in Fig. 7 and of M_{10} in Fig. 6. $\square$

Theorem 7. *The Möbius ladder M_n admits a ρ-tripartite labeling for all even $n \geq 4$.*

Proof. The cases $n \leq 10$ are covered in Lemma 6. We separate the rest of the proof into 3 cases.

Case 1 $n \equiv 0 \pmod 6$.

Let $n = 6t$. Thus, $|V(M_n)| = 12t$ and $|E(M_n)| = 18t$ where $t \geq 2$. Define a one-to-one function $f : V(M_{6t}) \to [0, 36t]$ as follows:

$$f(v_1) = 36t,$$
$$f(v_2) = 18t - 2,$$
$$f(v_i) = i, \quad v_i \in A_1 = \{v_i : i \text{ odd},\ 3 \leq i \leq 6t - 1\},$$
$$f(v_i) = 18t - 2i + 5, \quad v_i \in B_1 = \{v_i : i \text{ even},\ 4 \leq i \leq 4t - 2\},$$
$$f(v_i) = 18t - 2i - 1, \quad v_i \in B_2 = \{v_i : i \text{ even},\ 4t - 2 < i \leq 6t - 2\},$$
$$f(v_{6t}) = 36t - 1,$$
$$f(v_1') = 0,$$
$$f(v_2') = 36t - 2,$$
$$f(v_3') = 18t,$$
$$f(v_i') = i, \quad v_i' \in A_1' = \{v_i' : i \text{ even},\ 4 \leq i \leq 6t - 2\},$$
$$f(v_i') = 18t - 2i + 5, \quad v_i' \in B_1' = \{v_i' : i \text{ odd},\ 5 \leq i \leq 4t - 1\},$$
$$f(v_i') = 18t - 2i - 1, \quad v_i' \in B_2' = \{v_i' : i \text{ odd},\ 4t - 1 < i \leq 6t - 3\},$$

$$f(v'_{6t-1}) = 12t + 4,$$
$$f(v'_{6t}) = 6t + 2.$$

Note that $A = \{v'_1\} \cup A_1 \cup A'_1 \cup \{v'_{6t}\}$ and $B = \{v_2, v_6t, v'_3, v'_{6t-1}\} \cup B_1 \cup B_2 \cup B'_1 \cup B'_2$ and $C = \{v_1, v'_2\}$. Thus the domain of f is indeed $V(M_{6t})$. Next, we confirm that f is one-to-one. We compute

$$f(A_1) = \{3, 5, \ldots, 6t-1\},$$
$$f(A'_1) = \{4, 6, \ldots, 6t-2\},$$
$$f(B_1) = \{18t-3, 18t-7, \ldots, 10t+9\},$$
$$f(B_2) = \{10t-1, 10t-5, \ldots, 6t+3\},$$
$$f(B'_1) = \{18t-5, 18t-9, \ldots, 10t+7\},$$
$$f(B'_2) = \{10t-3, 10t-7, \ldots, 6t+5\}.$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(A) \subseteq [0, 6t+2]$ and $f(B \cup C) \subseteq [6t+3, 36t]$.

To help compute the edge labels, we will describe $f(M_{6t})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t} and P'_{6t} with their labels. We have $f(P_{6t}) = (36t, 18t-2, 3) + G_1 + G_2 + (6t-1, 36t-1)$, where

$$G_1 = \hat{P}(2(2t-2), 2, 4, 3, 10t+9),$$
$$G_2 = \hat{P}(2t, 2, 4, 4t-1, 6t+3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G_1)) &= \{6t+10+6i : 0 \le i \le 2t-3\} \cup \{6t+12+6i : 0 \le i \le 2t-3\}\\ &= \{\ell \equiv 4 \pmod 6 : 6t+10 \le \ell \le 18t-8\} \cup \{\ell \equiv 0 \pmod 6 : 6t+12 \le \ell \le 18t-6\},\\ \bar{f}(E(G_2)) &= \{4+6i : 0 \le i \le t-1\} \cup \{6+6i : 0 \le i \le t-1\}\\ &= \{\ell \equiv 4 \pmod 6 : 4 \le \ell \le 6t-2\} \cup \{\ell \equiv 0 \pmod 6 : 6 \le \ell \le 6t\}.\end{aligned}$$

Moreover, edge labels $18t+2$ and $18t-5$ occur on the path $(36t, 18t-2, 3)$ and the edge label $30t$ occurs on the edge $\{6t-1, 36t-1\}$.

Similarly, we have $f(P'_{6t}) = (0, 36t-2, 18t, 4) + G'_1 + G'_2 + (6t-2, 12t+4, 6t+2)$, where

$$G'_1 = \hat{P}(2(2t-2), 2, 4, 4, 10t+7),$$
$$G'_2 = \hat{P}(2(t-1), 2, 4, 4t, 6t+5).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{6t+7+6i : 0 \le i \le 2t-3\} \cup \{6t+9+6i : 0 \le i \le 2t-3\}\\ &= \{\ell \equiv 1 \pmod 6 : 6t+7 \le \ell \le 18t-11\} \cup \{\ell \equiv 3 \pmod 6 : 6t+7 \le \ell \le 18t-9\},\\ \bar{f}(E(G'_2)) &= \{7+6i : 0 \le i \le t-2\} \cup \{9+6i : 0 \le i \le t-2\}\\ &= \{\ell \equiv 1 \pmod 6 : 7 \le \ell \le 6t-5\} \cup \{\ell \equiv 3 \pmod 6 : 9 \le \ell \le 6t-3\}.\end{aligned}$$

Moreover, edge labels $36t-2$, $18t-2$ and $18t-4$ occur on the path $(0, 36t-2, 18t, 4)$ and edge labels $6t+6$ and $6t+2$ occur on the path $(6t-2, 12t+4, 6t+2)$.

For each spoke $\{v_i, v'_i\}$, the labels on spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 36t & \text{for } i = 1,\\ 18t & \text{for } i = 2,\\ 18t-3 & \text{for } i = 3,\\ 18t-3i+5 & \text{for } 4 \le i \le 4t-1,\\ 18t-3i-1 & \text{for } 4t-1 < i \le 6t-2,\\ 6t+5 & \text{for } i = 6t-1,\\ 30t-3 & \text{for } i = 6t.\end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = {} & \{\ell \equiv 2 \pmod 3 : 6t+8 \le \ell \le 18t-7\} \cup \{\ell \equiv 2 \pmod 3 : 5 \le \ell \le 6t-1\}\\ & \cup \{36t, 18t, 18t-3, 6t+5, 30t-3\}.\end{aligned}$$

Moreover, edge labels $30t-2$ and $36t-1$ occur on the edges $\{v_1, v'_{6t}\}$ and $\{v'_1, v_{6t}\}$.

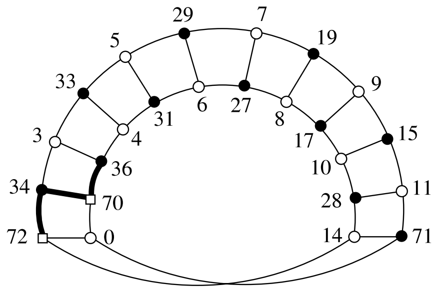


Fig. 8. A ρ-tripartite labeling of M_{12}.

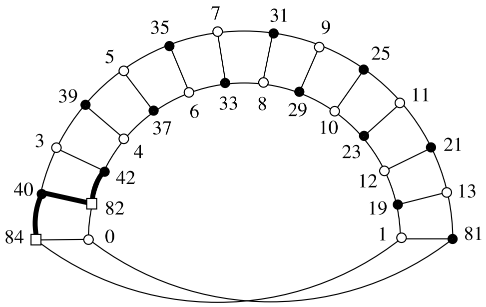


Fig. 9. A ρ-tripartite labeling of M_{14}.

It is easy to verify now that for each $\ell \in [1, 18t]$ either ℓ or $36t + 1 - \ell$ occurs on exactly one edge in M_{6t}. Hence the defined labeling is a ρ-labeling and condition (r1) for a ρ-tripartite labeling is satisfied. Condition (r2) also holds since $f(A) \subseteq [0, 6t + 2]$ and $f(B \cup C) \subseteq [6t + 3, 36t]$. Condition (r3) holds since $|f(v_1) - f(v_2)| + |f(v_2') - f(v_3')| = 36t$ and $|f(v_2) - f(v_2')| + |f(v_2) - f(v_2')| = 36t$, twice the number of edges of M_{6t}. Also $|f(b) - f(c)| = 36t$, where $b \in B$ and $c \in C$, is impossible since all vertices are in $[0, 36t]$ and $0 \in f(A)$. Thus condition (r4) holds, and we have a ρ-tripartite labeling of M_{6t}. Fig. 8 shows a ρ-tripartite labeling of M_{12}.

Case 2 $n \equiv 2 \pmod 6$.

Let $n = 6t + 2$, where $t \geq 2$. Thus, $|V(M_n)| = 12t + 4$ and $|E(M_n)| = 18t + 6$. Define a one-to-one function $f : V(M_{6t+2}) \to [0, 36t + 12]$ as follows:

$$\begin{aligned}
&f(v_1) = 36t + 12,\\
&f(v_2) = 18t + 4,\\
&f(v_i) = i, \quad v_i \in A_1,\ A_1 = \{v_i : i \text{ odd},\ 3 \leq i \leq 6t + 1\},\\
&f(v_i) = 18t - 2i + 11, \quad v_i \in B_1,\ B_1 = \{v_i : i \text{ even},\ 4 \leq i \leq 4t\},\\
&f(v_i) = 18t - 2i + 9, \quad v_i \in B_2,\ B_2 = \{v_i : i \text{ even},\ 4t < i \leq 6t\},\\
&f(v_{6t+2}) = 36t + 9,\\
&f(v_1') = 0,\\
&f(v_2') = 36t + 10,\\
&f(v_3') = 18t + 6,\\
&f(v_i') = i, \quad v_i' \in A_1',\ A_1' = \{v_i' : i \text{ even},\ 4 \leq i \leq 6t\},\\
&f(v_i') = 18t - 2i + 11, \quad v_i' \in B_1',\ B_1' = \{v_i' : i \text{ odd},\ 5 \leq i \leq 4t + 1\},\\
&f(v_i') = 18t - 2i + 9, \quad v_i' \in B_2',\ B_2' = \{v_i' : i \text{ odd},\ 4t + 1 < i \leq 6t + 1\},\\
&f(v_{6t+2}') = 1.
\end{aligned}$$

Note that $A = \{v_1', v_{6t+2}'\} \cup A_1 \cup A_1'$ and $B = \{v_2, v_{6t+2}, v_3'\} \cup B_1 \cup B_2 \cup B_1' \cup B_2'$ and $C = \{v_1, v_2'\}$. If we proceed as in case 1, it is easy to verify that we have a ρ-tripartite labeling of M_{6t+2}. Fig. 9 shows a ρ-tripartite labeling of M_{14}.

Case 3 $n \equiv 4 \pmod 6$.

Let $n = 6t - 2$, where $t \geq 3$. Thus, $|V(M_n)| = 12t - 4$ and $|E(M_n)| = 18t - 6$. Define a one-to-one function $f : V(M_{6t-2}) \to [0, 36t - 12]$ as follows:

$$f(v_1) = 36t - 12,$$

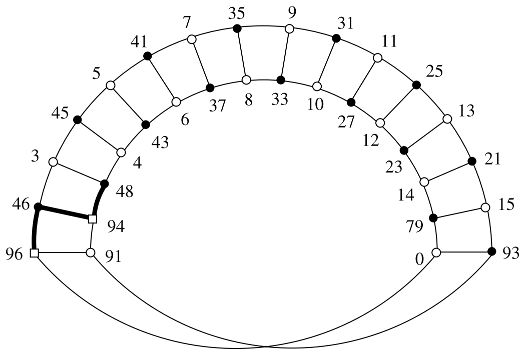


Fig. 10. A ρ-tripartite labeling of M_{16}.

$$f(v_2) = 18t - 8,$$
$$f(v_i) = i, \quad v_i \in A_1,\ A_1 = \{v_i : i \text{ odd},\ 3 \le i \le 6t - 3\},$$
$$f(v_i) = 18t - 2i - 1, \quad v_i \in B_1,\ B_1 = \{v_i : i \text{ even},\ 4 \le i \le 2t\},$$
$$f(v_i) = 18t - 2i - 3, \quad v_i \in B_2,\ B_2 = \{v_i : i \text{ even},\ 2t < i \le 4t - 2\},$$
$$f(v_i) = 18t - 2i - 5, \quad v_i \in B_3,\ B_3 = \{v_i : i \text{ even},\ 4t - 2 < i \le 6t - 4\},$$
$$f(v_{6t-2}) = 36t - 15,$$
$$f(v'_1) = 36t - 17,$$
$$f(v'_2) = 36t - 14,$$
$$f(v'_3) = 18t - 6,$$
$$f(v'_i) = i, \quad v_i \in A'_1,\ A'_1 = \{v_i : i \text{ even},\ 4 \le i \le 6t - 4\},$$
$$f(v'_i) = 18t - 2i - 1, \quad v'_i \in B'_1,\ B'_1 = \{v'_i : i \text{ odd},\ 5 \le i \le 2t - 1\},$$
$$f(v'_i) = 18t - 2i - 3, \quad v'_i \in B'_2,\ B'_2 = \{v'_i : i \text{ odd},\ 2t - 1 < i \le 4t - 3\},$$
$$f(v'_i) = 18t - 2i - 5, \quad v'_i \in B'_3,\ B'_3 = \{v'_i : i \text{ odd},\ 4t - 3 < i \le 6t - 5\},$$
$$f(v'_{6t-3}) = 30t - 11,$$
$$f(v'_{6t-2}) = 0.$$

Note that $A = \{v'_1, v'_{6t-2}\} \cup A_1 \cup A'_1$ and $B = \{v_2, v_{6t-2}, v'_3, v'_{6t-3}\} \cup B_1 \cup B_2 \cup B_3 \cup B'_1 \cup B'_2 \cup B'_3$ and $C = \{v_1, v'_2\}$. If we proceed as in case 1, it is easy to verify that we have a ρ-tripartite labeling of M_{6t-2}. Fig. 10 shows a ρ-tripartite labeling of M_{16}. □

3.3. ρ-tripartite labelings of tripartite cubic graphs of small order

According to the reference book *An Atlas of Graphs* [10], there are 27 connected cubic graphs of order at most 10. The complete graph K_4 is the only one of these graphs that is not tripartite. We give a ρ-tripartite labeling of each of the remaining 26 graphs in Table 1. We referenced these graphs in the same way they are referenced in [10]. Four of these graphs (C3, C8, C24, and C26) are bipartite. The ρ-tripartite labeling we show on the four bipartite graphs is an α-labeling. Thus we have the following.

Theorem 8. *Every tripartite cubic graph of order at most* 10 *admits a ρ-tripartite labeling.*

4. Concluding remarks

Because it is known that bipartite prisms and bipartite Möbius ladders admit α-labelings and in light of our results here, we have the following.

Corollary 9. *If G of size n is a prism, a Möbius ladder or a connected cubic graph of order at most* 10, *then there exists a cyclic G-decomposition of K_{2nt+1} for all positive integers t.*

We are currently investigating ρ-tripartite labelings of other cubic tripartite graphs such as cubic graphs of order 12 and generalized Petersen graphs. Based on our investigation, we believe that all connected tripartite cubic graphs admit ρ-tripartite labelings.

Conjecture 10. *Every connected tripartite cubic graph admits a ρ-tripartite labeling.*

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